ON THE CALCULATION OF CHARACTERISTIC VALUES BY THE BUBNOV-GALERKIN METHOD AND ITS APPLICATION IN THE THEORY OF HYDRODYNAMIC STABILITY

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In this paper we investigate two possibilities of improving the effectiveness of the Bubnov-Galerkin (*) method when applied to the characteristic value problems. The two methods are the following: the first one attempts this by improving the rate convergence of the method, while the other uses a numerical approach in a form suitable for computer processing, to the solution of the charcteristic equation. The results obtained are used to solve the problem of the stability of a plane Poiseuille flow.

1. Let us consider Equation

$$\varphi - \lambda K\varphi = 0 \tag{1.1}$$

in the Hilbert space H , where the operator K is completely continuous in H , and Equation

$$\varphi - \lambda K^*\varphi = 0 \tag{1.2}$$

which is a conjugate of (1.1).

We shall apply the B.-G. method to (1.1) and we shall take the system $\{u_k\}_1^\infty$, orthonormal in H , as a base set. Let λ_0 be an exact characteristic root of (1.1) and a simple pole of its resolvent, and let $\{\lambda_n\}$ represent approximate characteristic values of the above equation, the values being obtained by the B.-G. method and such that $\lambda_n \to \lambda_0$ when $n \to \infty$. By [1] we have the following estimate

$$|\lambda_n - \lambda_0| \leqslant C \max \left\{ \left\| u_0 - \sum_{k=1}^n (u_0, u_k) u_k \right\| \left\| v_0 - \sum_{k=1}^n (v_0, u_k) u_k \right\| \right\} \tag{1.3}$$

where C is a constant, and the maximum is taken over the whole of the normalized characteristic elements u_0 and v_0 of Equations (1.1) and (1.2), respectively, associated ith the characteristic values λ_0 and $\bar{\lambda}_0$, respectively. From (1.3) it follows that the accuracy of the nth approximation to the characteristic value is determined by the closeness of approach of characteristic functions and their projections on the linear envelope of the

*) Subsequently called the B.-G. method.

elements $u_1, \ldots, u_n$.

Let us now apply the B.-G. method to the determination of characteristic values of the boundary value problem of a linear differential equation. We know [2] that in many cases, the system of algebraic equations of the B.-G. method of the above problem is equivalent to the system of the B.-G. method for some equations of the type (1.1) in a suitably chosen Hilbert space H. This equivalence will assume a major importance in the proof of the convergence of the method. The inequality (1.3) shows that this equivalence can also be utilized in estimating the rapidity of convergence of characteristic values. Indeed, the latter is determined by the rapidity of convergence of the Fourier series for the characteristic functions over the base functions in the norm of H. As a rule we find, that the convergence in H is the convergence in mean together with the derivative up to some order.

Consider for example a boundary value problem for

$$(-1)^s \frac{d^{2s}\varphi}{dx^{2s}} - \lambda \sum_{i=0}^{2s-1} q_i(x) \frac{d^i\varphi}{dx^i} = 0 \tag{1.4}$$

where $q_i(x)$ are sufficiently smooth functions, and the boundary conditions are

$$\left.\frac{d^j\varphi}{dx^j}\right|_{x=a,\ x=b} = 0 \qquad (j = 0, 1, \ldots, s-1) \tag{1.5}$$

It can be shown [2] that the system of the B.-G. method for the problem (1.4), (1.5) over the set of base functions $\{u_k(x)\}$ satisfying (1.5) coincides with the system of the B.-G. method for the Fredholm homogeneous integral equation of the second kind with the continuous kernel, with respect to the function $d^s\varphi / dx^s$ over the base sequence $\{d^su_k / dx^s\}$. By (1.3), the functions $\{u_k\}$ should be selected in such a manner, that the Fourier series for $d^s\varphi/dx^s$ over the base $\{d^su_k / dx^s\}$ strongly converges in the mean.

In fact, such a selection is not always practical. Boundary functions in particular, in which the highest derivative has a small parameter, possess differential solutions which, as a rule, are much more complex, than the characteristic functions. Without the previous knowledge of the solution, it is very difficult to find a base sequence $\{u_k\}$, for which the Fourier series mentioned above, would converge rapidly. It is much easier to achieve a rapid convergence in the mean if the series is in terms of φ over $\{u_k\}$.

From this it follows that the approximate characteristic values will converge to exact values faster, if a slightly different form of the B.-G. method is used, namely, if the equivalent B.-G. system for the integral equation considers the characteristic function itself, rather than its derivatives. This can be achieved either by the prior reduction of the boundary value problem to the integral equation of the given type, or by applying inst ad of the classical B.-G. method, its generalization proposed by Petrov [3], directly to the boundary value problem. In the latter case, the equivalence condition leads to a definite relationship between the two bases of the B.-G.-Petrov method. We shall use the (1.4), (1.5) problem to illustrate this. Let $g(x,y)$ be a Green's function for the operator $(-1)^s d^{2s}\varphi / dx^{2s}$ with (1.5) valid. Coefficients of the algebraic system of equations of the B.-G.-Petrov method over the bases $\{u_k\}, \{v_k\}$ satisfying (1.5), will be of the form

$$\gamma_{jk}(\lambda) = \int_a^b \left[(-1)^s \frac{d^{2s}u_k}{dx^{2s}} - \lambda \sum_{i=1}^{2s-1} q_i(x) \frac{d^iu_k}{dx^i}\right] \bar{v}_j dx$$

Integrating by parts $2s$ times so as to reduce the order of the derivative in the first term and utilizing (1.5), we obtain

$$\gamma_{jk}(\lambda) = \int_a^b \left[u_k(x) - \sum_{i=1}^{2s-1} \int_a^b g(x, y)\, q_i(y) \frac{d^iu_k(y)}{dy^i}\, dy\right] (-1)^s \frac{d^{2s}v_j(x)}{dx^{2s}}\, dx$$

Next, utilizing the properties of the Green's function, we eliminate the derivatives of $u_k(y)$ integrating by parts the expressions contained within the square brackets. If now we put

$$v_j(x) = \int_a^b g(x, y)\, u_j(y)\, dy \qquad (1.6)$$

we shall see that $\gamma_{jk}(\lambda)$ are the coefficients of the system of the B.-G. method for the integral equation, referred to the function with the square integrable kernel over the base functions $\{u_k(x)\}$. Hence, Formula (1.6) defines the already mentioned relationship between the two bases.

2. When we apply the B.-G. method or the direct method to the problem on characteristic values, as a rule, reduces to the problem of determining the roots of a polynomial, which is given in the determinant form. If the order of this determnnant is sufficiently high and no information is available on the distribution of its roots in the complex plane, then the process of root-finding becomes very involved. Determination of the coefficients of the polynomial poses the chief difficulty. A method suitable for computer processing is given below.

Let $\Delta(\lambda)$, be a determinant, the elements of which are functions of the complex parameter λ, and which can be calculated for any value of λ. We assume that this determinant is an nth degree polynomial in

$$\Delta(\lambda) = p_0 + p_1\lambda + p_2\lambda^2 + \ldots + p_n\lambda^n \qquad (2.1)$$

To find its coefficients p_k $(k = 0, 1, \ldots, n)$, we denote by λ_j $(j = 0, 1, \ldots, n)$ all the solutions of the equation $\lambda^{n+1} = 1$, i.e.

$$\lambda_j = \exp\left(i\,\frac{2\pi j}{n+1}\right) \qquad (j = 0, 1, \ldots, n) \qquad (2.2)$$

Obviously, $\lambda_j = \lambda_1{}^j$. If we now put $\Delta_j = \Delta(\lambda_j)$ $(j = 0, 1, \ldots, n)$, then the coefficients p_k are defined by

$$\sum_{k=0}^{n} p_k \lambda_j{}^k = \Delta_j \qquad (j = 0, 1, \ldots, n) \qquad (2.3)$$

with Vandermond's matrix which is difficult to invert numerically. The system (2.3) with the nodes λ_j arbitrarily chosen, is known in the interpolation theory [4], but Lagrange's, Newton's and other interpolation formulas are not found to be more suitable for calculating p_k, than the direct solution of (2.3).

We shall now show that when the nodes are chosen according to Formula (2.2), the system (2.3) possesses an exact general solution for any n and Δ_j, and the formulas for p_k are simple and suitable for programing. Let us multiply the jth equation of (2.3) by λ_1^{-js} where s is an integer. On rearranging the resulting equations, we obtain

$$\sum_{k=0}^{n} p_k \sum_{j=0}^{n} \lambda_1{}^{j(k-s)} = \sum_{j=0}^{n} \lambda_1{}^{-js}\, \Delta_j \qquad (2.4)$$

Its left-hand side is a sum of $n + 1$ terms of a geometrical progression, hence using (2.2), we obtain

$$\sum_{j=0}^{n} \lambda_1{}^{j(k-s)} = (n+1)\,\delta_{ks}$$

where δ_{ks} is a Kronecker delta. From (2.4) it now follows that

$$p_s = \frac{1}{n+1} \sum_{j=0}^{n} \lambda_1{}^{-js} \Delta_j = \frac{1}{n+1} \sum_{j=0}^{n} \bar{\lambda}_s{}^j \Delta_j \qquad (s = 0, 1, \ldots, n) \qquad (2.5)$$

where a superscript $\bar{}$ denotes a complex conjugate.

When (2.5) is used to work out the values of p_s ,

$$\bar{\lambda}_s = \cos\frac{2\pi s}{n+1} - i\sin\frac{2\pi s}{n+1} \qquad (s = 0, 1, \ldots, n)$$

should be used in the following manner: for p_1 all the terms are used, for p_2 every second term is used, for p_3 every third term is used, and so on.

A formula analogous to (2.5) is easily obtained if all the solutions of the equation $\lambda^{n+1} = R^{n+1}$, where $R > 0$ are taken as nodes. In this case

$$p_s = \frac{1}{R^s(n+1)}\sum_{j=0}^{n} \bar{\lambda}_s^{\,j}\,\Delta(R\lambda_j) \qquad (s = 0, 1, \ldots, n) \tag{2.6}$$

where λ_s is again defined by (2.2).

3. Let us now consider a problem on the linear stability of a plane Poiseuille flow with respect to plane perturbations, where the stream function is symmetrical with respect to the axis of the channel. We know [5] that this problem can be reduced to the problem on characteristic values for Equation

$$\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi - i\alpha R\,[(1 - x^2 - c)(\varphi'' - \alpha^2\varphi) + 2\varphi] = 0 \tag{3.1}$$

with the boundary conditions

$$\varphi'(0) = \varphi'''(0) = \varphi(1) = \varphi'(1) = 0 \tag{3.2}$$

Here $\varphi(x)$ and α are the complex amplitude and the wave number of the perturbations, respectively; $i = \sqrt{-1}$; R is the Reynolds number; $c = c_r + ic_i$ is a parameter associated with the development of perturbations with time. The problem under discussion is set in terms of this parameter. All the magnitudes are dimensionless, lengths are expressed in terms of semi-width of the channel and the velocities in terms of maximum velocity of the basic stream.

The above problem has already been investigated [5 and 7]. In particular, the authors of [7] used the B.-G. method and the base functions recommended by [3]; small part of the neutral curve was calculated with an acceptable accuracy only in the 20-th approximation.

To check the formulas developed in Sections 2 and 3 of this paper, we shall use two methods for the problem (3.1), (3.2.).

1. B.-G. method with the base function set consisting of polynomials of even power, satisfying the boundary conditions. To make integration easier, we shall write these functions as

$$u_k(x) = \frac{P_{2k+2}(x)}{4k+3} - \frac{2(4k+1)P_{2k}(x)}{(4k-1)(4k+3)} + \frac{P_{2k-2}(x)}{4k-1} \tag{3.3}$$

where

$$P_{2k}(x) = \frac{1}{2^{2k}(2k)!}\,\frac{d^{2k}}{dx^{2k}}(x^2-1)^{2k} \qquad (k = 1, 2, \ldots) \tag{3.4}$$

are Legendre polynomials. It is easy to show that the method converges if the system $\{u_k''\}$ is complete in the mean in the class of functions orthogonal to a constant and square integrable over the interval (0.1).

It is easily seen that (3.3) possesses the above properties.

2. B.-G. method with the same base functions, but applied to the integral equation in $\varphi(x)$ equivalent to (3.1),(3.2) or, in other words, the B.-G. method with two sets of base functions namely $\{u_k(x)\}$ and $\{v_k(x)\}$, first of which is obtained from (3.3), while the second one from the condition

$$v_k^{IV} = u_k, \quad v_k'(0) = v_k'''(0) = v_k(1) = v_k'(1) = 0$$

The completeness in the mean of the function $\{u_k\}$ over the interval (0.1) is the sufficient condition for the convergence of the method, and is fulfilled here.

Table 1

n	c		Δc	
	c_r	c_i	Δc_r	Δc_i
5	0.2837	0.0199	—	—
6	0.2542	0.0295	—0.0295	0.0017
7	0.2386	0.0155	—0.0156	—0.0061
8	0.2341	0.0062	—0.0045	—0.0093
9	0.2383	0.0046	0.0042	—0.0016
10	0.2364	0.0050	—0.0019	0.0004

Table 2

R / α		2500		10 000	
		c_r	c_i	c_r	c_i
0.9	$n=9$	0.2856	—0.0215	0.2270	0.0048
	$n=10$	0.2857	—0.0211	0.2253	0.0050
	T	0.2857	—0.0212	0.2261	0.0040
1.0	$n=9$	0.3011	—0.0145	0.2382	0.0046
	$n=10$	0.3011	—0.0141	0.2365	0.0050
	T	0.3011	—0.0142	0.2375	0.0037
1.1	$n=9$	0.3148	—0.0112	0.2475	0.0001
	$n=10$	0.3148	—0.0106	0.2457	0.0018
	T	0.3148	—0.0108	0.2470	—0.0003

Characteristic determinants obtained by the above methods were calculated using the method shown in Section 3 up to and including the 10th approximation. Roots of the resulting polynomials were found by the method of parabolas [8], or Newton's method was used where required to improve accuracy of the final results. Calculations were performed on a computer.

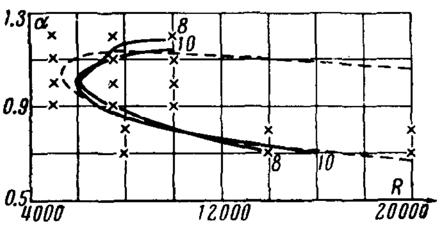

Fig. 1

In the first series of calculations, characteristic values of c for $\alpha = 1$ and $R = 10000$ were obtained in order to compare the methods 1 and 2, and to compare the results with those of Thomas [6]

Method 1 did not lead to any sequence of characteristic values possessing a usable convergence. Method 2 on the other hand gave (see Table 1) a characteristic value with the relative accuracy less than 1%. Table 2 shows the calculated characteristic values against those of Thomas [6] (the latter denoted by T).

The aim of the second series of calculations was to investigate how the value of R infuences the rate of convergence of method 2. With $R = 100$, $\alpha = 1$, first 10 approximations gave first five characteristic numbers with the relative accuracy of about 0.1%, while $R = 1000$, $\alpha = 1$, gave three characteristic numbers.

The last series of results was used to construct a neutral curve $c_i(\alpha, R) = 0$ in the 8th and 10th approximations according to method 2 (see Fig.1). Characteristic values were calculated for maximum values of c_i at the points α, R of the plane as shown on the figure. Points of the neutral curve were obtained by interpolation. For comparison, a neutral curve obtained by Lin' [5] is also shown on the graph.

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